Exercise 3.3.10

If $f(x) = \begin{cases} x^2 & x < 0 \\ e^{-x} & x > 0 \end{cases}$, what are the even and odd parts of $f(x)$?

Solution

Any function $f(x)$ can be written as

$$\begin{aligned} f(x) &= \frac{f(x)}{2} + \frac{f(-x)}{2} + \frac{f(x)}{2} - \frac{f(-x)}{2} \\ &= \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}. \end{aligned}$$

This first fraction is the even part because swapping x with $-x$ doesn't change it, and this second fraction is the odd part because swapping x with $-x$ gives the same fraction with a minus sign. For the prescribed function, we have

$$\text{Even Part:} \quad \begin{cases} \frac{1}{2}[x^2 + e^{-(-x)}] & x < 0 \\ \frac{1}{2}[e^{-x} + (-x)^2] & x > 0 \end{cases} = \begin{cases} \frac{1}{2}(x^2 + e^x) & x < 0 \\ \frac{1}{2}(e^{-x} + x^2) & x > 0 \end{cases}$$

$$\text{Odd Part:} \quad \begin{cases} \frac{1}{2}[x^2 - e^{-(-x)}] & x < 0 \\ \frac{1}{2}[e^{-x} - (-x)^2] & x > 0 \end{cases} = \begin{cases} \frac{1}{2}(x^2 - e^x) & x < 0 \\ \frac{1}{2}(e^{-x} - x^2) & x > 0 \end{cases}.$$